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Short Communication

Strongly nonlinear vibrations of damped oscillators with two nonsmooth limits

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Abstract

A family of strongly nonlinear oscillators with a generalized power form elastic force and viscous damping is considered. An explicit analytical solution is obtained as a combination of smooth and nonsmooth functions. Two different nonsmooth functions involved into the solution are associated with two different nonsmooth limits of the oscillator as the exponent becomes either zero or infinity. As a result, the solution is drastically simplified to give the best match with numerical tests if approaching any of the two limits.

Strongly nonlinear oscillators with power-form elastic terms have been considered for a long time, see for instance Refs. [1–18]. Major efforts focused, however, on temporal mode shapes of periodic vibrations and the amplitude–frequency response.

This paper suggests an approximate solution for damped vibrations that captures two nonsmooth limits of the restoring force characteristic given by the potential energy of the generalized power form [1,2]

$$P(x) = |x|^{\alpha+1}/(\alpha+1), \tag{1}$$

where α is considered as *any* real number from the interval $0 \leqslant \alpha < \infty$ due to the operation of absolute value of x.

Within the set of real numbers, the exponent α can be continuously moved to either zero or infinity, where the potential energy (1) takes different kind of nonsmooth shapes as illustrated by Fig. 1. The corresponding force $\mathrm{d}P(x)/\mathrm{d}x$ has a finite discontinuity at $x = 0$ if $\alpha = 0$ and an infinite discontinuity at $x = \pm 1$ if $\alpha = \infty$. Note that, under no damping condition, vibrations of both limit oscillators are described by elementary functions such as piecewise-parabolic or piecewise-linear periodic functions.

Let us consider now the differential equation of motion of the oscillator with linear damping

$$\ddot{x} + 2\mu\dot{x} + \mathrm{sgn}(x)|x|^{\alpha} = 0, \tag{2}$$

where overdots mean derivatives with respect to time, t.

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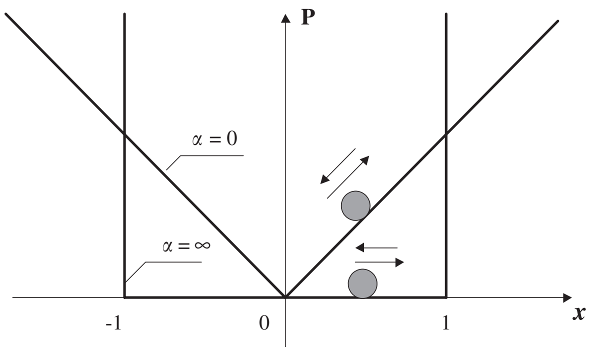

Fig. 1. Potential energy representation for the two limit oscillators.

The approximate analytical solution of Eq. (2) will be derived from the truncated series of successive approximations for the undamped case, $\mu = 0$ [2]

$$x(\tau) = A\,\mathrm{sgn}(\tau)\left[|\tau| - \frac{|\tau|^{\alpha+2}}{\alpha+2} - \frac{\alpha}{2(\alpha+2)}\left(\frac{|\tau|^{\alpha+2}}{\alpha+2} - \frac{|\tau|^{2\alpha+3}}{2\alpha+3}\right) - R_3 - R_4 - \cdots\right], \tag{3}$$

where $\tau = \tau(\varphi)$ and $\varphi = \varphi(t)$ are defined such that $\tau(\varphi) = (2/\pi)\arcsin\sin(\pi\varphi/2)$ is the triangular wave of the period $T = 4$ with respect to the phase variable φ whose temporal rate is

$$\dot{\varphi} = \omega = \frac{A^{(\alpha-1)/2}}{\sqrt{\alpha+1}}\left\{1 + \frac{\alpha}{2(\alpha+2)}\left[1 + \frac{(\alpha+1)^2}{(\alpha+2)(2\alpha+3)}\right] + r_3 + r_4 + \cdots\right\}^{-1/2} \tag{4}$$

and expressions

$$0 < R_i(\alpha, |\tau|) < \frac{\alpha|\tau|^{\alpha+2}}{2^{i-1}(\alpha+2)^2},$$

$$0 < r_i(\alpha) < \frac{\alpha}{2^i(\alpha+2)} \tag{5}$$

estimate those terms that will be neglected below.

Estimates (5) indicate that series (3) and (4) converge slowly. However, it will be shown that asymptotics of large and small exponents α essentially improve precision of the truncated series even though first few terms of the series are included.

Further, in addition to the truncation, a μ-perturbation technique is used in order to adapt solution (3) for the case of nonzero damping. Note that, for oscillators (2), whether or not the damping is small depends on the level of amplitude and exponent α. By assuming that the influence of damping is negligible during one cycle of vibration, one can apply expressions (3) and (4) to estimating the magnitudes of damping and elastic forces. As a result, the condition of small damping derives in the form

$$\mu^2 \ll \tfrac{1}{4}(\alpha+1)A^{\alpha-1}. \tag{6}$$

Now, following the idea of parameter variation, let us assume that $A = A(\mu t)$ and thus $\omega = \omega(\mu t)$. Then, as a first-order asymptotic with respect to μ, one obtains [3]

$$2\omega(A' + A) + \omega' A = 0, \tag{7}$$

where $' \equiv \mathrm{d}/\mathrm{d}(\mu t)$.

Eqs. (4) and (7) admit exact solution

$$A = C\exp\left(-\frac{4\mu t}{\alpha+3}\right), \quad \varphi = \varphi_\infty\left[1 - \exp\left(-2\mu\frac{\alpha-1}{\alpha+3}t\right)\right], \tag{8}$$

where C is an arbitrary constant, and

$$\varphi_{\infty} = \frac{1}{2\mu}\frac{\alpha+3}{\alpha-1}\frac{C^{(\alpha-1)/2}(\alpha+2)\sqrt{4\alpha+6}}{\sqrt{(\alpha+1)(7\alpha^3+31\alpha^2+47\alpha+24)}}. \tag{9}$$

As follows from expressions (8) and (9), the linear system $\alpha = 1$ plays the role of a boundary between the two strongly nonlinear areas

$$N_0 = \{\alpha : 0 \leqslant \alpha \ll 1\} \quad \text{and} \quad N_{\infty} = \{\alpha : 1 \ll \alpha < \infty\}. \tag{10}$$

In other words, it will be shown below that the magnitude $\alpha = 1$ separates two qualitatively different regions of the dynamics determined by the influence of different nonsmooth limits of the potential well. In particular, if $\alpha > 1$ then the phase variable φ is bounded by its finite value φ_{∞} as $t \to \infty$. In contrast, if $\alpha < 1$ then the phase and thus its temporal rate are exponentially growing as the amplitude decays and the system approaches the bottom of the potential well. The physical meaning of this effect is most clear from the limit case $\alpha \to 0$, see the discussion below.

Figs. 2–5 illustrate solution (3)–(9) for large and small exponents α, respectively, and suggest quite a good match with numerical solution in *both* branches of the exponent (10). The numerical solutions shown by dashed lines were obtained by the standard solver *NDSolve* built in *Mathematica*. Fig. 2 also shows that some divergence between the curves occurs when the amplitude is decreased to the level about $A = 0.6$. Below this level, the condition of small damping (6) is not guaranteed any more. In contrast, the curves are in a better match for smaller amplitudes if $\alpha < 1$, see Fig. 4. In this case, the amplitude decay just strengthens condition (6). The phase plane diagrams shown in Figs. 3 and 5 have qualitatively different shapes as dictated by the

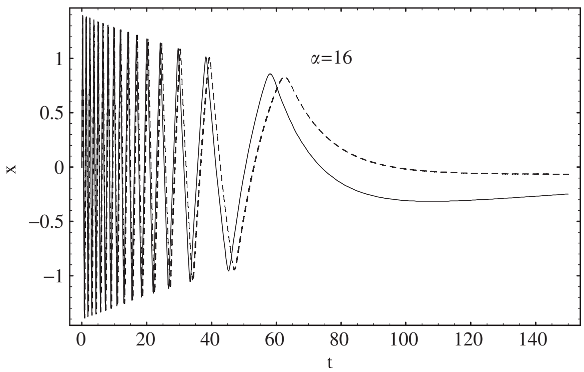

Fig. 2. Temporal mode shape of the vibration for $\alpha \in N_{\infty}$, $C = 1.5$ and $\mu = 0.04$; here and below, the dashed line represents numerical solutions.

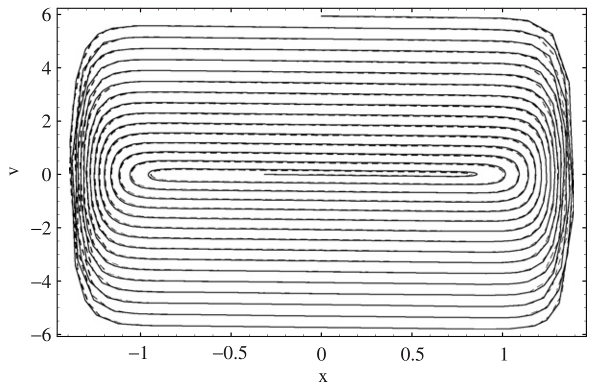

Fig. 3. Phase plane diagram for $\alpha \in N_{\infty}$; $v = \dot{x}$.

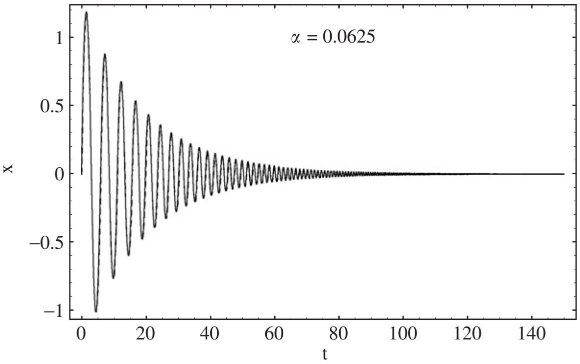

Fig. 4. Temporal mode shape of the vibration for $\alpha \in N_0$, $C = 2.5$ and $\mu = 0.04$.

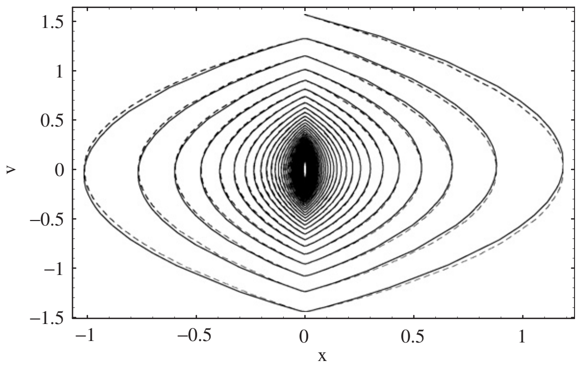

Fig. 5. Phase plane diagram for $\alpha \in N_0$; $v = \dot{x}$.

influence of different nonsmooth limits of the potential well, see Fig. 1. Let us show now that solution (3) captures both nonsmooth limits $\alpha \longrightarrow 0$ and ∞.

For physically meaningful transition to the limits, let us express the arbitrary parameter C through the initial velocity $v_0 = \dot{x}|_{t=0}$,

$$C = \left[\frac{v_0^2(\alpha+1)(7\alpha^3+31\alpha^2+47\alpha+24)}{2(\alpha+2)^2(2\alpha+3)}\right]^{1/(\alpha+1)}$$

and consider the two different cases.

(1) As $\alpha \longrightarrow \infty$, the solution (3)–(9) gives

$$x = \tau(\varphi), \quad \varphi = \frac{v_0}{2\mu}[1 - \exp(-2\mu t)]. \tag{11}$$

Solution (11) *exactly* describes the system motion in the square potential well.

(2) When $\alpha \longrightarrow 0$, expressions (3)–(9) reduce to

$$x = v_0^2 \exp\left(-\frac{4}{3}\mu t\right)\tau(\varphi)\left(1 - \frac{|\tau(\varphi)|}{2}\right), \quad \varphi = \frac{3}{2\mu v_0}\left[\exp\left(\frac{2}{3}\mu t\right) - 1\right], \tag{12}$$

where the identity $\mathrm{sgn}[\tau(\varphi)]|\tau(\varphi)| \equiv \tau(\varphi)$ has been taken into account.

If, in addition $\mu = 0$, then solution (12) also *exactly* describes the system dynamics with another nonsmooth limit of the potential energy, $|x|$, as shown in Fig. 1. However, if $\mu \neq 0$ then substituting solution (12) into the differential equation of motion gives an error $O(\mu^2)$. In terms of first-order asymptotic solutions, the error of order μ occurs on the time period of order $1/\mu$. Therefore, solution (12) exactly captures the carrying shape of the vibration, but gives only asymptotic estimate for the exponential decay.

Note that the error of solution (3) is due to the error of the iterative procedure for elastic vibrations and the error of asymptotic for energy dissipation. As shown above, the error of successive approximations vanishes as either $\alpha \longrightarrow \infty$ or 0, but the error of asymptotic vanishes only as $\alpha \longrightarrow \infty$.

Finally, let us discuss the qualitative difference of the dynamics in the parameter intervals N_0 and N_∞. As follows from Eq. (8), for $\alpha \in N_0$, the phase of vibration and the corresponding frequency are exponentially increasing in the slow time scale μt. In the limit case $\alpha = 0$, according to solution (12), the amplitude and frequency are, respectively

$$A(\mu t) = \frac{v_0^2}{2}\exp\left(-\frac{4}{3}\,\mu t\right) \quad \text{and} \quad \dot{\varphi} = \frac{1}{v_0}\exp\left(\frac{2}{3}\,\mu t\right) = \frac{1}{\sqrt{2A(\mu t)}}. \tag{13}$$

Expressions (13) describe increasingly rapid vibrations—‘dither’—near the corner of the potential energy $|x|$ as the amplitude approaches zero. In contrast, when $\alpha \in N_\infty$, the oscillator makes a limited number of cycles such that the phase φ remains bounded for any time t. Again, the most clear interpretation is obtained in the limit case $\alpha \to \infty$, when, as follows from (11), the phase variable $\varphi(t)$ represents the total distance passed by the particle by time t, and $\dot{\varphi} = v$ is the absolute value of the velocity. Since the barriers are perfectly elastic the particle reflects with no energy loss, the velocity $v(t)$ remains continuous function of time described by the linear differential equation $\dot{v} + 2\mu v = 0$ or $\ddot{\varphi} + 2\mu\dot{\varphi} = 0$. Under the initial conditions $\varphi(0) = 0$ and $\dot{\varphi}(0) = v_0$, one obtains exactly solution (11).

In conclusion, the explicit analytical solution for a class of strongly nonlinear oscillators with viscous damping is introduced. Two different nonsmooth functions involved into the solution are associated with two different nonsmooth limits of the oscillator. As a result, the solution is drastically simplified to give the best match with numerical tests if approaching any of the two limits.

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